MATH 022 Exam #1B Solutions
Given on September 17, 2009
Show all non-trivial calculations.

1. Factor the expressions.

   (a) $4x^2 - 15x + 9$

   $$4x^2 - 15x + 9 = (4x-3)(x-3)$$

   (b) $16x^2 + 72x + 81$

   Note that $16x^2 = (4x)^2$ and $81 = 9^2$. This suggests that this *may* be a perfect square trinomial. If it is, then we can use the formula $a^2 + 2ab + b^2 = (a+b)^2$ to factor it. We must have $a = 4x$ and $b = 9$, and we must verify that the crossterm is of the form $2ab$. Since

   $$72x = 2(4x)(9) = 2ab$$

   the expression is indeed a perfect square trinomial and factors as

   $$16x^2 + 72x + 81 = (4x+9)^2$$

   (c) $6x^3 + 21x^2 - 45x$

   $$6x^3 + 21x^2 - 45x = 3x(2x^2 + 7x - 15) = 3x(2x-3)(x+5)$$

   (d) $3x^3 - 2x^2 - 12x + 8$

   $$\begin{aligned} 3x^3 - 2x^2 - 12x + 8 &= x^2(3x-2) - 4(3x-2) \\ &= (3x-2)(x^2-4) \\ &= (3x-2)(x+2)(x-2) \end{aligned}$$

   (e) $8y^3 - 27$

Note that $8y^3 = (2y)^3$ and $27 = 3^3$. This is a difference of cubes. We can use the formula $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ to factor it.

$$\begin{aligned} 8y^3 - 27 &= (2y)^3 - 3^3 \\ &= (2y-3)[(2y)^2 + (2y)(3) + (3)^2] \\ &= (2y-3)(4y^2 + 6y + 9) \end{aligned}$$

2. Solve the equations.

(a) $3(2x-3) + 2x = 5x + 9$

$$\begin{aligned} 3(2x-3) + 2x &= 5x + 9 \\ 6x - 9 + 2x &= 5x + 9 \\ 8x - 9 &= 5x + 9 \\ 8x - 9 - 5x + 9 &= 5x + 9 - 5x + 9 \\ 3x &= 18 \\ \frac{3x}{3} &= \frac{18}{3} \\ x &= 6 \end{aligned}$$

(b) $2x^2 - 5x + 3 = 0$

$$\begin{aligned} 2x^2 - 5x + 3 &= 0 \\ (2x-3)(x-1) &= 0 \end{aligned}$$

$$\begin{aligned} 2x - 3 &= 0 & \text{or} \qquad x - 1 &= 0 \\ 2x - 3 + 3 &= 0 + 3 & x - 1 + 1 &= 0 + 1 \\ 2x &= 3 & x &= 1 \\ 2x/2 &= 3/2 & & \\ x &= 3/2 & & \end{aligned}$$

(c) $(3x-4)(x+1) = 6$

$$\begin{aligned}
(3x-4)(x+1) &= 6\\
3x^2-x-4 &= 6\\
3x^2-x-4-6 &= 6-6\\
3x^2-x-10 &= 0\\
(3x+5)(x-2) &= 0
\end{aligned}$$

$$\begin{aligned}
3x+5 &= 0\\
3x+5-5 &= 0-5\\
3x &= -5\\
\tfrac{3x}{3} &= \tfrac{-5}{3}\\
x &= -\tfrac{5}{3}
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
x-2 &= 0\\
x-2+2 &= 0+2\\
x &= 2
\end{aligned}$$

(d) $5x^2 = 2x$

$$\begin{aligned}
5x^2 &= 2x\\
5x^2-2x &= 2x-2x\\
x(5x-2) &= 0
\end{aligned}$$

$$x = 0 \quad \text{or} \quad
\begin{aligned}
5x-2 &= 0\\
5x-2+2 &= 0+2\\
5x &= 2\\
\tfrac{5x}{5} &= \tfrac{2}{5}\\
x &= \tfrac{2}{5}
\end{aligned}$$

3. Reduce the rational expressions to lowest terms.

(a) $\dfrac{x^2-8x+16}{2x^2-8x}$

$$\begin{aligned}
\frac{x^2-8x+16}{2x^2-8x} &= \frac{(x-4)^2}{2x(x-4)}\\
&= \frac{x-4}{2x}
\end{aligned}$$

(b) $\dfrac{x^2+3x-4}{x^2+6x+8}$

$$\begin{aligned}\frac{x^2+3x-4}{x^2+6x+8} &= \frac{(x-1)(x+4)}{(x+4)(x+2)} \\ &= \frac{x-1}{x+2}\end{aligned}$$

4. Multiply or divide the following expressions. Be sure your answer is reduced to simplest form.

(a) $\dfrac{x^2}{x-5} \cdot \dfrac{x^2-25}{x}$

$$\begin{aligned}\frac{x^2}{x-5} \cdot \frac{x^2-25}{x} &= \frac{x^2}{x-5} \cdot \frac{(x+5)(x-5)}{x} \\ &= x(x+5)\end{aligned}$$

(b) $\dfrac{4x-10}{x^2+3x} \cdot \dfrac{x^2-9}{6x-15}$

$$\begin{aligned}\frac{4x-10}{x^2+3x} \cdot \frac{x^2-9}{6x-15} &= \frac{2(2x-5)}{x(x+3)} \cdot \frac{(x+3)(x-3)}{3(2x-5)} \\ &= \frac{2(x-3)}{3x}\end{aligned}$$

(c) $\dfrac{3y^2}{2x} \div \dfrac{6y}{4x^2}$

$$\frac{3y^2}{2x} \div \frac{6y}{4x^2} = \frac{3y^2}{2x} \cdot \frac{4x^2}{6y} = xy$$

(d) $\dfrac{2x^2+3x-2}{2x-1} \div \dfrac{x-2}{4x^2-1}$

$$\begin{aligned}\frac{2x^2+3x-2}{2x-1} \div \frac{x-2}{4x^2-1} &= \frac{2x^2+3x-2}{2x-1} \cdot \frac{4x^2-1}{x-2} \\ &= \frac{(2x-1)(x+2)}{2x-1} \cdot \frac{(2x-1)(2x+1)}{x-2} \\ &= \frac{(x+2)(2x-1)(2x+1)}{x-2}\end{aligned}$$

5. Add or subtract as indicated. Be sure your answer is reduced to simplest form.

(a) $\frac{5}{x} - \frac{x}{3}$

$$\begin{aligned} \frac{5}{x} - \frac{x}{3} &= \frac{5}{x} \cdot \frac{3}{3} - \frac{x}{3} \cdot \frac{x}{x} \\ &= \frac{15}{3x} - \frac{x^2}{3x} \\ &= \frac{15 - x^2}{3x} \end{aligned}$$

(b) $\frac{2x}{x+4} + \frac{8}{x+4}$

$$\begin{aligned} \frac{2x}{x+4} + \frac{8}{x+4} &= \frac{2x+8}{x+4} \\ &= \frac{2(x+4)}{x+4} \\ &= 2 \end{aligned}$$

(c) $\frac{x}{x-4} - \frac{8x}{x^2-16}$

$$\begin{aligned} \frac{x}{x-4} - \frac{8x}{x^2-16} &= \frac{x}{x-4} \cdot \frac{x+4}{x+4} - \frac{8x}{(x+4)(x-4)} \\ &= \frac{x(x+4)}{(x+4)(x-4)} - \frac{8x}{(x+4)(x-4)} \\ &= \frac{x(x+4) - 8x}{(x+4)(x-4)} \\ &= \frac{x^2 + 4x - 8x}{(x+4)(x-4)} \\ &= \frac{x^2 - 4x}{(x+4)(x-4)} \\ &= \frac{x(x-4)}{(x+4)(x-4)} \\ &= \frac{x}{x+4} \end{aligned}$$

(d) $\dfrac{x+6}{x^2-9} - \dfrac{x+3}{x^2-2x-3}$

$$\begin{aligned}
\frac{x+6}{x^2-9} - \frac{x+3}{x^2-2x-3} &= \frac{x+6}{(x+3)(x-3)} - \frac{x+3}{(x-3)(x+1)} \\
&= \frac{x+6}{(x+3)(x-3)} \cdot \frac{x+1}{x+1} - \frac{x+3}{(x-3)(x+1)} \cdot \frac{x+3}{x+3} \\
&= \frac{(x+6)(x+1)}{(x+3)(x-3)(x+1)} - \frac{(x+3)(x+3)}{(x+3)(x-3)(x+1)} \\
&= \frac{(x+6)(x+1)-(x+3)(x+3)}{(x+3)(x-3)(x+1)} \\
&= \frac{x^2+7x+6-(x^2+6x+9)}{(x+3)(x-3)(x+1)} \\
&= \frac{x-3}{(x+3)(x-3)(x+1)} \\
&= \frac{1}{(x+3)(x+1)}
\end{aligned}$$

6. Solve the equations.

(a) $\dfrac{2x}{3} - \dfrac{3}{2} = 2$

$$\begin{aligned}
6 \times \frac{2x}{3} - 6 \times \frac{3}{2} &= 6 \times 2 \\
4x - 9 &= 12 \\
4x - 9 + 9 &= 12 + 9 \\
4x &= 21 \\
\frac{4x}{4} &= \frac{21}{4} \\
x &= \frac{21}{4}
\end{aligned}$$

Note: Because we are solving an equation here, we can multiply both sides by the LCD. We could *not* do this for any of the problems in #5 because in those we were not solving equations, but simplifying expressions.

(b) $2 - \dfrac{1}{x} = \dfrac{6}{x^2}$

$$\text{LCD} = \text{x}^2$$

$$\begin{aligned}
2 \cdot x^2 - \frac{1}{x} \cdot x^2 &= \frac{6}{x^2} \cdot x^2 \\
2x^2 - x &= 6 \\
2x^2 - x - 6 &= 6 - 6 \\
2x^2 - x - 6 &= 0 \\
(2x+3)(x-2) &= 0
\end{aligned}$$

$$\begin{aligned}
2x + 3 &= 0 & \text{or} \qquad x - 2 &= 0 \\
2x + 3 - 3 &= 0 - 3 & x - 2 + 2 &= 0 + 2 \\
2x &= -3 & x &= 2 \\
\tfrac{2x}{2} &= -\tfrac{3}{2} \\
x &= -\tfrac{3}{2}
\end{aligned}$$

(c) $\dfrac{3}{x+1} + \dfrac{x}{x-1} = \dfrac{2}{x^2-1}$

$$\text{LCD} = (\text{x} + 1)(\text{x} - 1)$$

$$\begin{aligned}
\frac{3}{x+1} \cdot (x+1)(x-1) + \frac{x}{x-1} \cdot (x+1)(x-1) &= \frac{2}{(x+1)(x-1)} \cdot (x+1)(x-1) \\
3(x-1) + x(x+1) &= 2 \\
3x - 3 + x^2 + x &= 2 \\
x^2 + 4x - 3 &= 2 \\
x^2 + 4x - 3 - 2 &= 2 - 2 \\
x^2 + 4x - 5 &= 0 \\
(x+5)(x-1) &= 0
\end{aligned}$$

$$\begin{aligned}
x + 5 &= 0 & \text{or} \qquad x - 1 &= 0 \\
x + 5 - 5 &= 0 - 5 & x - 1 + 1 &= 0 + 1 \\
x &= -5 & x = 1
\end{aligned}$$

$x = 1$ is extraneous, since it gives us undefined expressions. The only solution is $x = -5$.

(d) $\dfrac{1}{x^2+x-2} + \dfrac{3}{x^2+3x+2} = \dfrac{2}{x^2-1}$

Factoring the denominators, this is

$$\frac{1}{(x+2)(x-1)} + \frac{3}{(x+2)(x+1)} = \frac{2}{(x+1)(x-1)}$$

$$\text{LCD} = (\text{x}+2)(\text{x}+1)(\text{x}-1)$$

$$\frac{1}{(x+2)(x-1)} \cdot (x+2)(x+1)(x-1) + \frac{3}{(x+2)(x+1)} \cdot (x+2)(x+1)(x-1)$$

$$= \frac{2}{(x+1)(x-1)} \cdot (x+2)(x+1)(x-1)$$

$$\begin{aligned}
(x+1) + 3(x-1) &= 2(x+2) \\
x + 1 + 3x - 3 &= 2x + 4 \\
4x - 2 &= 2x + 4 \\
4x - 2 - 2x + 2 &= 2x + 4 - 2x + 2 \\
2x &= 6 \\
\frac{2x}{2} &= \frac{6}{2} \\
x &= 3
\end{aligned}$$

7. Carol walks 2 miles on a slight downhill grade. She then takes a one mile shortcut uphill back to her starting point. She walks 2mph slower on the uphill portion than she walks on the downhill portion. If the entire trip takes her one hour, what is her walking speed on the downhill portion?

Let x = her downhill walking rate. Then her uphill walking rate is 2 mph slower, or $x-2$. We use the formula $t = d/r$ to complete the table below.

	d	r	t
downhill	2	x	$\frac{2}{x}$
uphill	1	$x-2$	$\frac{1}{x-2}$

Since it takes her one hour to complete the trip, we deduce the equation

$$\frac{2}{x} + \frac{1}{x-2} = 1$$

The least common denominator is $x(x-2)$. We multiply each term in the equation by this.

$$\begin{aligned}
x(x-2)\cdot\frac{2}{x} + x(x-2)\cdot\frac{1}{x-2} &= x(x-2)\cdot 1 \\
2(x-2) + x &= x(x-2) \\
2x - 4 + x &= x^2 - 2x \\
3x - 4 &= x^2 - 2x \\
3x - 4 - 3x + 4 &= x^2 - 2x - 3x + 4 \\
x^2 - 5x + 4 &= 0 \\
(x-4)(x-1) &= 0
\end{aligned}$$

$$\begin{aligned}
x - 4 &= 0 & \text{or} \qquad x - 1 &= 0 \\
x - 4 + 4 &= 0 + 4 & x - 1 + 1 &= 0 + 1 \\
x &= 4 & x &= 1
\end{aligned}$$

We can rule out the solution $x = 1$, since that would make her walking speed on the uphill portion negative, which makes no sense. Then her speed on the downhill portion is 4 mph.

8. A swimming pool can be filled with two inlets. The first inlet can fill the pool in 7 hours by itself. The second inlet can fill the pool in 5 hours by itself. How long will it take to fill the pool using both inlets?

Let $x =$ the time it takes to fill the pool with both inlets open.

	time	pool/hr
inlet 1	7	1/7
inlet 2	5	1/5
both	x	1/x

The portion of the pool filled in one hour by both inlets is the sum of the portions filled by each of the two inlets. This gives us

$$\begin{aligned}
\frac{1}{7}+\frac{1}{5} &= \frac{1}{x}\\
\frac{1}{7}\cdot 35x+\frac{1}{5}\cdot 35x &= \frac{1}{x}\cdot 35x\\
5x+7x &= 35\\
12x &= 35\\
\frac{12x}{12} &= \frac{35}{12}
\end{aligned}$$

It will take $\frac{35}{12}$ hours or 2 hours and 55 minutes to fill the pool with both inlets.

9. Simplify the complex fractions.

(a) $\dfrac{\dfrac{2x^2}{x+5}}{\dfrac{4x}{x^2-25}}$

$$\begin{aligned}
\frac{\dfrac{2x^2}{x+5}}{\dfrac{4x}{x^2-25}} &= \frac{2x^2}{x+5}\cdot\frac{x^2-25}{4x}\\
&= \frac{2x^2}{x+5}\cdot\frac{(x+5)(x-5)}{4x}\\
&= \frac{x(x-5)}{2}
\end{aligned}$$

(b) $\dfrac{\dfrac{1}{2y}-5}{2-\dfrac{1}{y}}$

$$\frac{\dfrac{1}{2y}-5}{2-\dfrac{1}{y}} = \frac{\dfrac{1}{2y}-5}{2-\dfrac{1}{y}}\cdot\frac{2y}{2y}$$

$$= \frac{\frac{1}{2y} \cdot 2y - 5 \cdot 2y}{2 \cdot 2y - \frac{1}{y} \cdot 2y}$$

$$= \frac{1 - 10y}{4y - 2}$$

10. If it takes me 4 hours to grade 25 exams, how long should it take me to grade 40 of the same exams?

    Let $x =$ the time to grade 40 exams. Then we have

$$\begin{aligned} \frac{4}{25} &= \frac{x}{40} \\ 4 \cdot 40 &= 25x \\ 160 &= 25x \\ \frac{160}{25} &= \frac{25x}{25} \\ x &= \frac{160}{25} = \frac{32}{5} \end{aligned}$$

    It should take 6.4 hours or 6 hours 24 minutes to grade 40 exams.