

1. Find the exact value (not the calculator approximation) of the following trigonometric expressions. Justify your answers.

(a) $\cos \frac{7\pi}{12}$

Note that $\frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$. We use the sum formula.

$$\begin{aligned}\cos \frac{7\pi}{12} &= \cos \left(\frac{\pi}{4} + \frac{\pi}{3} \right) \\ &= \cos \frac{\pi}{4} \cos \frac{\pi}{3} - \sin \frac{\pi}{4} \sin \frac{\pi}{3} \\ &= \left(\frac{1}{\sqrt{2}} \right) \left(\frac{1}{2} \right) - \left(\frac{\sqrt{3}}{2} \right) \left(\frac{1}{\sqrt{2}} \right) \\ &= \frac{1 - \sqrt{3}}{2\sqrt{2}} \\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

(We could just as easily use the half-angle formula, since $\frac{7\pi}{12} = \frac{1}{2} \cdot \frac{7\pi}{6}$. There are choices here.)

(b) $\tan \frac{\pi}{12}$

Note that $\frac{\pi}{12} = \frac{1}{2} \cdot \frac{\pi}{6}$. We use the half-angle formula.

$$\begin{aligned}\tan \frac{\pi}{12} &= \tan \left(\frac{1}{2} \cdot \frac{\pi}{6} \right) \\ &= \frac{1 - \cos \frac{\pi}{6}}{\sin \frac{\pi}{6}} \\ &= \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} \\ &= \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} \cdot \frac{2}{2} \\ &= 2 - \sqrt{3}\end{aligned}$$

(c) $\sin 165^\circ$

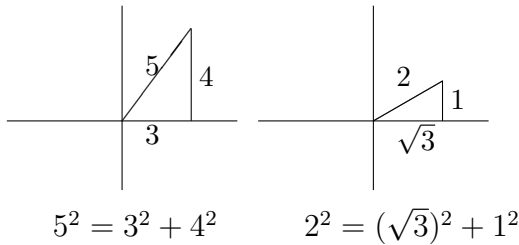
Note that $165^\circ = 120^\circ + 45^\circ$. We will use the sum formula.

$$\begin{aligned}\sin 165^\circ &= \sin (120^\circ + 45^\circ) \\ &= \sin 120^\circ \cos 45^\circ + \cos 120^\circ \sin 45^\circ \\ &= \left(\frac{\sqrt{3}}{2} \right) \left(\frac{1}{\sqrt{2}} \right) + \left(-\frac{1}{2} \right) \left(\frac{1}{\sqrt{2}} \right)\end{aligned}$$

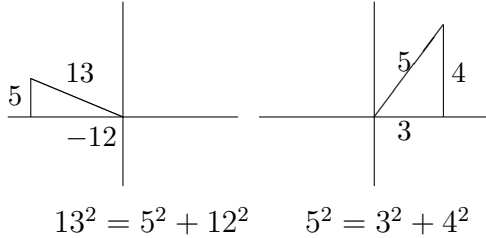
$$\begin{aligned}
&= \frac{\sqrt{3}-1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

(d) $\cos\left(\cos^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2}\right)$

$$\begin{aligned}
\cos\left(\cos^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2}\right) &= \cos\left(\cos^{-1}\frac{3}{5}\right)\cos\left(\sin^{-1}\frac{1}{2}\right) \\
&\quad + \sin\left(\cos^{-1}\frac{3}{5}\right)\sin\left(\sin^{-1}\frac{1}{2}\right) \\
&= \left(\frac{3}{5}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{2}\right) \\
&= \frac{3\sqrt{3}+4}{10}
\end{aligned}$$



2. Given that $\sin \alpha = \frac{5}{13}$ $\frac{\pi}{2} < \alpha < \pi$ $\cos \beta = \frac{3}{5}$ $0 < \beta < \frac{\pi}{2}$, find



(a) $\sin(\alpha + \beta)$

$$\begin{aligned}
\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
&= \left(\frac{5}{13}\right)\left(\frac{3}{5}\right) + \left(-\frac{12}{13}\right)\left(\frac{4}{5}\right) \\
&= \frac{15 - 48}{65} \\
&= -\frac{33}{65}
\end{aligned}$$

(b) $\cos(\alpha - \beta)$

$$\begin{aligned}
\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
&= \left(-\frac{12}{13}\right)\left(\frac{3}{5}\right) + \left(\frac{5}{13}\right)\left(\frac{4}{5}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{-36 + 20}{65} \\
&= -\frac{16}{65}
\end{aligned}$$

(c) $\tan(\beta - \alpha)$

$$\begin{aligned}
\tan(\beta - \alpha) &= \frac{\tan \beta - \tan \alpha}{1 + \tan \beta \tan \alpha} \\
&= \frac{\left(\frac{4}{3}\right) - \left(-\frac{5}{12}\right)}{1 + \left(\frac{4}{3}\right)\left(-\frac{5}{12}\right)} \cdot \frac{36}{36} \\
&= \frac{48 + 15}{36 - 20} \\
&= \frac{63}{16}
\end{aligned}$$

3. Verify three of the five identities. Select only three and check the appropriate box for the three you wish graded.

(a) Grade. $\frac{\cot^2 \theta}{1 + \cot^2 \theta} = \cos^2 \theta$

$$\begin{aligned}
\frac{\cot^2 \theta}{1 + \cot^2 \theta} &= \frac{\frac{\cos^2 \theta}{\sin^2 \theta}}{1 + \frac{\cos^2 \theta}{\sin^2 \theta}} \cdot \frac{\sin^2 \theta}{\sin^2 \theta} \\
&= \frac{\cos^2 \theta}{\sin^2 \theta + \cos^2 \theta} \\
&= \frac{\cos^2 \theta}{1} = \cos^2 \theta
\end{aligned}$$

(b) Grade. $\frac{1}{2} \sin 2\theta \sec \theta = \sin \theta$

$$\begin{aligned}
\frac{1}{2} \sin 2\theta \sec \theta &= \frac{1}{2} (2 \sin \theta \cos \theta) \sec \theta \\
&= \sin \theta \cos \theta \frac{1}{\cos \theta} \\
&= \sin \theta
\end{aligned}$$

(c) Grade. $\frac{\cos(\alpha - \beta)}{\cos \alpha \cos \beta} = 1 + \tan \alpha \tan \beta$

$$\begin{aligned}
\frac{\cos(\alpha - \beta)}{\cos \alpha \cos \beta} &= \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\cos \alpha \cos \beta} \\
&= 1 + \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta} \\
&= 1 + \tan \alpha \tan \beta
\end{aligned}$$

(d) \square Grade. $(1 + \cos \theta) \left(\tan \frac{\theta}{2} \right) = \sin \theta$

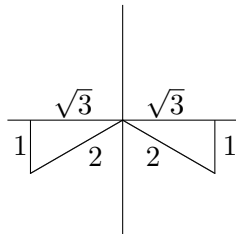
$$\begin{aligned} (1 + \cos \theta) \left(\tan \frac{\theta}{2} \right) &= (1 + \cos \theta) \left(\frac{1 - \cos \theta}{\sin \theta} \right) \\ &= \frac{1 - \cos^2 \theta}{\sin \theta} \\ &= \frac{\sin^2 \theta}{\sin \theta} \\ &= \sin \theta \end{aligned}$$

(e) \square Grade. $\csc \theta - \sin \theta = \cos \theta \cot \theta$

$$\begin{aligned} \csc \theta - \sin \theta &= \frac{1}{\sin \theta} - \sin \theta \\ &= \frac{1}{\sin \theta} - \frac{\sin^2 \theta}{\sin \theta} \\ &= \frac{1 - \sin^2 \theta}{\sin \theta} \\ &= \frac{\cos^2 \theta}{\sin \theta} \\ &= \cos \theta \left(\frac{\cos \theta}{\sin \theta} \right) \\ &= \cos \theta \cot \theta \end{aligned}$$

4. Find all solutions to the following equations on the interval $0 \leq \theta < 2\pi$.

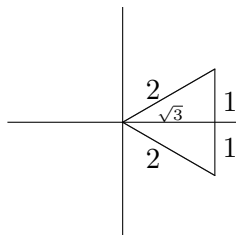
(a) $\sin \theta = -\frac{1}{2}$



The reference angle is $\frac{\pi}{6}$. The angles measured in standard position are

$$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \quad \theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

(b) $\cos 2\theta = \frac{\sqrt{3}}{2}$



The reference angle is $\frac{\pi}{6}$.

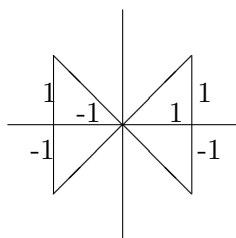
$$2\theta = \pm \frac{\pi}{6} + 2n\pi$$

$$\theta = \pm \frac{\pi}{12} + n\pi$$

In the interval $[0, 2\pi)$ this gives us the angles

$$\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$$

(c) $\tan^2 \theta = 1$



$$\tan^2 \theta = 1$$

$$\tan \theta = \pm \sqrt{1} = \pm 1$$

As shown in the picture, there are 4 angles,

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

(d) $2 \cos^2 \theta + \cos \theta = 1$

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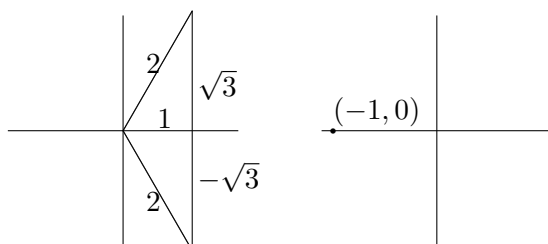
$$2 \cos^2 \theta + \cos \theta - 1 = 0$$

$$(2 \cos \theta - 1)(\cos \theta + 1) = 0$$

$$2 \cos \theta - 1 = 0 \quad \text{or} \quad \cos \theta + 1 = 0$$

$$\cos \theta = \frac{1}{2} \quad \cos \theta = -1$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3} \quad \theta = \pi$$



$$(e) \sin 2\theta + 2 \sin \theta = 0$$

$$\begin{aligned}\sin 2\theta + 2 \sin \theta &= 0 \\ 2 \sin \theta \cos \theta + 2 \sin \theta &= 0 \\ 2 \sin \theta (\cos \theta + 1) &= 0 \\ \sin \theta = 0 &\text{ or } \cos \theta + 1 = 0 \\ \theta = 0, \pi &\quad \cos \theta = -1 \\ &\quad \theta = \pi\end{aligned}$$

The solutions are $\theta = 0, \pi$.